

PHYSICS 225 HOMEWORK 3

①



4-vectors $\pi: (E_\pi \ 0 \ 0 \ p_\pi)$ $E_\pi = \sqrt{p_\pi^2 + m_\pi^2}$

$\mu: (E_\mu \ p_\mu \sin \alpha \ 0 \ p_\mu \cos \alpha)$ $E_\mu = \sqrt{p_\mu^2 + m_\mu^2}$

$e: (E_e \ -E_e \sin \theta \ 0 \ E_e \sin \theta)$ $m_e = 0$

$E - \vec{p}$ conservation

$$\begin{cases} E_\pi = E_\mu + E_e & (3) \\ p_\mu \sin \alpha = E_e \sin \theta & (1) \\ p_\mu \cos \alpha = p_\pi - E_e \cos \theta & (2) \end{cases}$$

$$(1)^2 + (2)^2 \rightarrow p_\mu^2 = E_e^2 \sin^2 \theta + p_\pi^2 + E_e^2 \cos^2 \theta - 2p_\pi E_e \cos \theta$$

$$p_\mu^2 = E_e^2 + p_\pi^2 - 2p_\pi E_e \cos \theta \quad (4)$$

$$(3) \rightarrow E_\mu = E_\pi - E_e$$

$$E_\mu^2 = E_\pi^2 + E_e^2 - 2E_\pi E_e$$

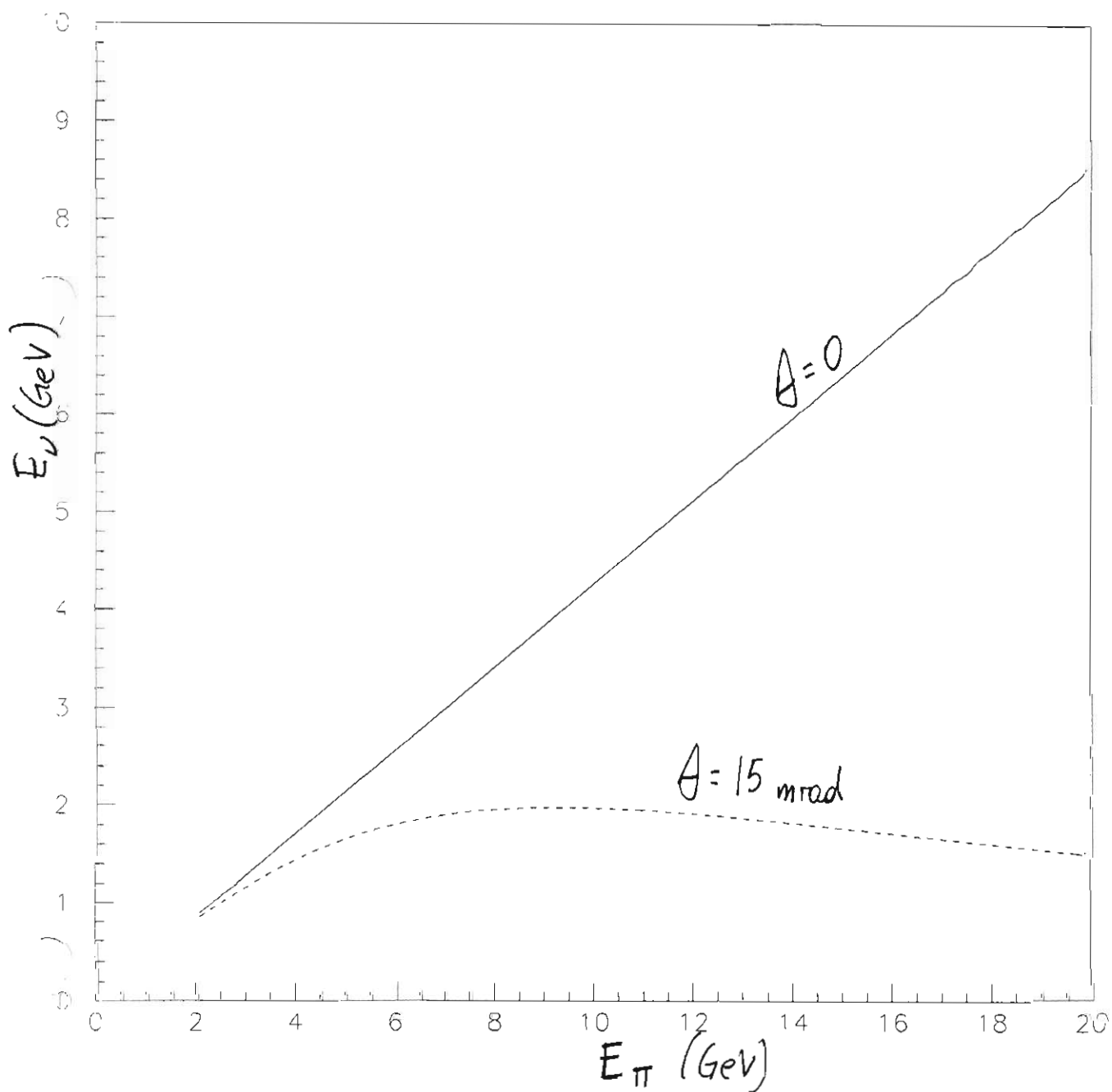
$$p_\mu^2 + m_\mu^2 = E_\pi^2 + E_e^2 - 2E_\pi E_e$$

Use eqn (4) for p_μ^2

$$m_\mu^2 + E_e^2 + p_\pi^2 - 2p_\pi E_e \cos \theta = E_e^2 + \cancel{p_\pi^2} + E_\pi^2 \cancel{p_\pi^2} - 2E_\pi E_e$$

$$2E_e (E_\pi - p_\pi \cos \theta) = E_\pi^2 - p_\pi^2 - m_\mu^2 = m_\pi^2 - m_\mu^2$$

$$E_e = \frac{1}{2} \frac{m_\pi^2 - m_\mu^2}{(E_\pi - p_\pi \cos \theta)}$$



② $\frac{\Delta m^2 L}{4E}$ should be a number

This has unit of $[E] \cdot [L]$

Note $\hbar c = 0.197 \text{ GeV fm} = 0.197 \cdot 10^{-18} \text{ GeV km}$

$\Rightarrow$ If I measure Δm^2 in GeV, E in GeV, and L in km, I find

$$\begin{aligned} \frac{\Delta m^2 L}{4E} &= \frac{1}{4} \frac{\Delta m^2 (\text{GeV}^2)}{E (\text{GeV})} \frac{L (\text{km})}{0.197 \cdot 10^{-18}} \\ &= 1.27 \cdot 10^{18} \frac{\Delta m^2 (\text{GeV}^2)}{E (\text{GeV})} L (\text{km}) \\ &= 1.27 \frac{\Delta m^2 (\text{eV}^2) L (\text{km})}{E (\text{GeV})} \quad \checkmark \end{aligned}$$

3

(i)

4

Using equation (15) of hep-ex/0506165

$$P(\bar{\nu}_e \rightarrow \bar{\nu}_\beta) = -4 \sum_{i>j} \text{Real}(U_{ei}^* U_{\beta i} U_{ej} U_{\beta j}^*) \sin^2 \Delta_{ij} \\ - 2 \sum_{i>j} \text{Im}(U_{ei}^* U_{\beta i} U_{ej} U_{\beta j}^*) \sin 2\Delta_{ij}$$

with $\beta = \mu$ or τ . Then the prob of disappearance is $P = P(\bar{\nu}_e \rightarrow \bar{\nu}_\mu) + P(\bar{\nu}_e \rightarrow \bar{\nu}_\tau)$

$$P = -4 \sum_{i>j} \text{Real}[U_{ei}^* U_{ej} (U_{\mu i} U_{\mu j}^* + U_{\tau i} U_{\tau j}^*)] \sin^2 \Delta_{ij} \\ - 2 \sum_{i>j} \text{Im}[U_{ei}^* U_{ej} (U_{\mu i} U_{\mu j}^* + U_{\tau i} U_{\tau j}^*)] \sin 2\Delta_{ij}$$

Unitarity says $U_{ei} U_{ej}^* + U_{\mu i} U_{\mu j}^* + U_{\tau i} U_{\tau j}^* = \delta_{ij}$
Multiply by $U_{ei}^* U_{ej}$, to obtain (for $i \neq j$)

$$|U_{ei} U_{ej}^*|^2 + U_{ei}^* U_{ej} (U_{\mu i} U_{\mu j}^* + U_{\tau i} U_{\tau j}^*) = 0$$

$$\Rightarrow U_{ei}^* U_{ej} (U_{\mu i} U_{\mu j}^* + U_{\tau i} U_{\tau j}^*) = -|U_{ei} U_{ej}^*|^2$$

Plug this into equation (1)

$$P = -4 \sum_{i>j} \text{Real}[-|U_{ei} U_{ej}^*|^2] \sin^2 \Delta_{ij} \\ - 2 \sum_{i>j} \text{Im}[-|U_{ei} U_{ej}^*|^2] \sin 2\Delta_{ij}$$

The imaginary part is obviously zero

Problem 4

(a)

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} e^{+i\phi_1/2} & 0 & 0 \\ 0 & e^{i\phi_2/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(equation 40 in Boris Kayser's notes)

Then

$$m_{\beta\beta}^2 = \left| \sum_i m_i U_{ei}^2 \right| = \left| c_{12}^2 c_{13}^2 e^{i\phi_1} m_1 + s_{12}^2 c_{13}^2 e^{i\phi_2} m_2 + s_{13}^2 e^{-2i\delta} m_3 \right|$$

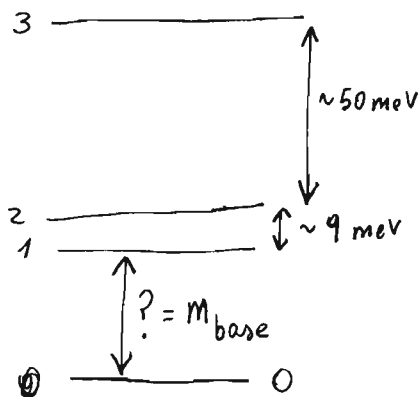
Clearly this depends on the Majorana phases ϕ_1 & ϕ_2

$$(b) \quad \Delta m_{31}^2 \sim 2.4 \cdot 10^{-3} \text{ eV}^2 \quad \Delta m_{12}^2 \sim 8 \cdot 10^{-5} \text{ eV}^2$$

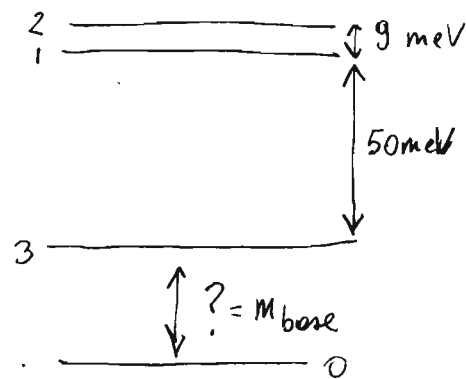
$$\sqrt{\Delta m_{31}^2} \sim 50 \text{ meV} \quad \sqrt{\Delta m_{12}^2} \sim 9 \text{ meV}$$

Two possibilities:

NORMAL



Inverted



From part (a), with $\Delta\phi = \phi_2 - \phi_1$ and $\rho = \delta + \phi$,

$$M_{\beta\beta}^2 = \left| C_{12}^2 C_{13}^2 m_1 + S_{12}^2 C_{13}^2 e^{i\Delta\phi} m_2 + S_{13}^2 e^{i\rho} m_3 \right|^2$$

We know that $\sin^2 2A_{13} \leq 0.12$, $\sin^2 A_{13} \leq 0.03$

$$\Rightarrow C_{13} \approx 1$$

$$M_{\beta\beta}^2 \approx \left| C_{12}^2 m_1 + S_{12}^2 e^{i\Delta\phi} m_2 + S_{13}^2 e^{i\rho} m_3 \right|^2$$

$$C_{12}^2 \approx \cos^2 34 \approx 0.7$$

$$S_{12}^2 \approx 0.3$$

↑
small

In the "normal" case, m_3 large, I can arrange for the three terms in the sum to be of the same order - Then I can pick $\Delta\phi \neq \rho$ to have both Imaginary and real parts of the sum = 0
eg suppose $S_{13}^2 \sim 0.01$, then I can pick $m_3 \sim 100 m_1$
for example $m_1 \sim 0.5 \text{ meV}$ and $m_3 \sim 50 \text{ meV}$
So I can make $M_{\beta\beta}$ as small as I like -
In the "inverted" case, $m_3 < m_1, m_2 \Rightarrow$ drop the third term in sum

$$M_{\beta\beta}^2 \approx \left| C_{12}^2 m_1 + S_{12}^2 m_2 e^{i\Delta\phi} \right|^2$$

$$M_{\beta\beta}^2 \approx C_{12}^4 m_1^4 + S_{12}^4 m_2^4 \cos^2 \Delta\phi + 2 C_{12}^2 S_{12}^2 m_1 m_2 \cos \Delta\phi$$

$$M_{\beta\beta}^2 \approx C_{12}^4 m_1^4 + S_{12}^4 m_2^4 + \frac{1}{2} \sin^2 2A_{12} \cos \Delta\phi$$

Trig identities $C^4 = C^2(1 - S^2) = C^2 - S^2 C^2$

$$S^4 = S^2(1 - C^2) = S^2 - S^2 C^2$$

$$M_{\beta\beta}^2 = C_{12}^2 M_1^2 + S_{12}^2 M_2^2 - S_{12}^2 C_{12}^2 (M_1^2 + M_2^2) + \frac{1}{2} \sin^2 2A_{12} \cos \Delta\phi M_1 M_2$$

$$M_{\beta\beta}^2 = C_{12}^2 M_1^2 + S_{12}^2 M_2^2 - \frac{1}{4} \sin^2 2A_{12} (M_1^2 + M_2^2) + \frac{1}{2} \sin^2 2A_{12} \cos \Delta\phi M_1 M_2$$

$$\begin{cases} M_1 = M_0 - \delta \\ M_2 = M_0 + \delta \end{cases} \quad \begin{aligned} M_1 \cdot M_2 &= M_0^2 - \delta^2 \\ M_1^2 &= M_0^2 + \delta^2 - 2M_0\delta \\ M_2^2 &= M_0^2 + \delta^2 + 2M_0\delta \\ M_1^2 + M_2^2 &= 2(M_0^2 + \delta^2) \end{aligned}$$

$$M_{\beta\beta}^2 = M_0^2 + \delta^2 + 2M_0\delta (\sin^2 A_{12} - \cos^2 A_{12}) - \frac{1}{2} \sin^2 2A_{12} (M_0^2 + \delta^2) + \frac{1}{2} \sin^2 2A_{12} \cos \Delta\phi (M_0^2 - \delta^2)$$

$$M_{\beta\beta}^2 = \cancel{\cos^2 2A_{12}}$$

$$M_{\beta\beta}^2 = M_0^2 \left[1 - \sin^2 2A_{12} \left(\frac{1 - \cos \Delta\phi}{2} \right) \right] + \delta^2 \left[1 - \sin^2 2A_{12} \left(\frac{1 + \cos \Delta\phi}{2} \right) \right] + 2M_0\delta \sin 2A_{12}$$

$$\text{But } \frac{1 - \cos \Delta\phi}{2} = \sin^2 \frac{\Delta\phi}{2} \quad \frac{1 + \cos \Delta\phi}{2} = \cos^2 \frac{\Delta\phi}{2}$$

$$M_{\beta\beta}^2 = M_0^2 \left(1 - \sin^2 2A_{12} \sin^2 \frac{\Delta\phi}{2} \right) + \underbrace{\delta^2 \left(1 - \sin^2 2A_{12} \cos^2 \frac{\Delta\phi}{2} \right)}_{\text{positive}} + \underbrace{2M_0\delta \sin 2A_{12}}_{\text{positive}}$$

~~$M_{\beta\beta}^2$~~ for $\delta \ll m_0 \leftarrow$

$$M_{\beta\beta} \approx m_0 \sqrt{1 - \sin^2 2A_{12} \sin^2 \frac{\Delta\phi}{2}}$$

$$M_{\beta\beta} \gtrsim m_0 \cos 2A_{12}$$

Definitely true for the inverted mass hierarchy since $\delta \sim 9 \text{ meV}$ and $m_0 \gtrsim 55 \text{ meV} !!$

Note the δ and $\sqrt{m_0\delta}$ correction to this are all positive

5

hep-ph/0506165

~~Depends~~ eqn 10 says

$$(a) A_{\alpha\beta} = \text{Amp}(\nu_\alpha \rightarrow \nu_\beta) = \sum_i U_{\alpha i}^* U_{\beta i} \exp -i \left[m_i^2 \frac{L}{2E} \right]$$

$$|A_{\alpha\beta}|^2 = \sum_{i,j} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* \exp -i \left[\frac{(m_i^2 - m_j^2) L}{2E} \right]$$

$$\text{Let } \Delta_{ij} = (m_i^2 - m_j^2) \frac{L}{2E}$$

Rewrite it as

$$|A_{\alpha\beta}|^2 = \sum_i U_{\alpha i}^* U_{\beta i} U_{\alpha i} U_{\beta i}^* + 2 \sum_{i < j} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* e^{-i \Delta_{ij}} \quad (1)$$

But unitarity says

$$\sum_i U_{\alpha i}^* U_{\beta i} = \delta_{\alpha\beta}$$

Take the square of this

$$\sum_{i,j} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* = \delta_{\alpha\beta}$$

$$\sum_i U_{\alpha i}^* U_{\beta i} U_{\alpha i} U_{\beta i}^* + 2 \sum_{i < j} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* = \delta_{\alpha\beta}$$

$$\sum_i U_{\alpha i}^* U_{\beta i} U_{\alpha i} U_{\beta i}^* = \delta_{\alpha\beta} - 2 \sum_{i < j} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*$$

Substitute into equation (1)

$$|A_{\alpha\beta}|^2 = \delta_{\alpha\beta} - 2 \sum_{i < j} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* (1 - e^{-i\Delta_{ij}})$$

$$|A_{\alpha\beta}|^2 = \delta_{\alpha\beta} - 2 \sum_{i < j} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* [1 - \cos \Delta_{ij} + i \sin \Delta_{ij}]$$

This expression is real by construction, so clearly

$$|A_{\alpha\beta}|^2 = \text{Re}(|A_{\alpha\beta}|^2)$$

$$= \delta_{\alpha\beta} - 2 \sum_{i < j} \text{Re}(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*) (1 - \cos \Delta_{ij}) \\ + 2 \sum_{i < j} \text{Im}(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*) \sin \Delta_{ij}$$

But $1 - \cos \Delta_{ij} = 2 \sin^2 \frac{\Delta_{ij}}{2}$

$$\Delta_{ij} = \frac{(m_i^2 - m_j^2)L}{2E}$$

$$|A_{\alpha\beta}|^2 = \delta_{\alpha\beta} - 4 \sum_{i < j} \text{Re}(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*) \sin^2 \frac{\Delta_{ij}}{2} + 2 \sum_{i < j} \text{Im}(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*) \sin \Delta_{ij}$$

(b) We see from equation 40 in hep-ph/0506165 that the Majorana phases change the matrix as

$$U \rightarrow U \cdot \begin{pmatrix} e^{i\alpha_1/2} & 0 & 0 \\ 0 & e^{i\alpha_2/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow$ the only elements that are affected are

~~$$U_{\alpha 1} \rightarrow U_{\alpha 1} e^{i\alpha_1/2}$$~~

$$U_{\alpha 1} \rightarrow U_{\alpha 1} e^{i\alpha_1/2}$$

$$U_{\alpha 2} \rightarrow U_{\alpha 2} e^{i\alpha_2/2} \quad \forall \alpha$$

but in the equation for $|A_{\alpha\beta}|^2$ we always have products $U_{\alpha i}^* U_{\beta i}$ and $U_{\beta j} U_{\alpha j}^*$ which are invariant

~~since the (for $\alpha = 1$, for example) $U_{12}^* U_{12}$ and $U_{13}^* U_{13}$ are invariant~~

these Majorana phases have absolutely no effect on $|A_{\alpha\beta}|^2$

(c) Show that $\text{Im}(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*)$ is the same for all i & j , up to a sign (for $\alpha \neq \beta$) (and $i \neq j$)

Unitarity:

$$U_{\alpha 1}^* U_{\beta 1} + U_{\alpha 2}^* U_{\beta 2} + U_{\alpha 3}^* U_{\beta 3} = 0 \quad (2)$$

Multiply eqn (2) by $U_{\alpha 2} U_{\beta 2}^*$

$$U_{\alpha 1}^* U_{\beta 1} U_{\alpha 2} U_{\beta 2}^* + \underbrace{|U_{\alpha 2} U_{\beta 2}|^2}_{\text{REAL !!}} + U_{\alpha 2} U_{\beta 2}^* U_{\alpha 3} U_{\beta 3}^* = 0$$

$$\Rightarrow \boxed{\text{Im}(U_{\alpha 1}^* U_{\beta 1} U_{\alpha 2} U_{\beta 2}^*) = + \text{Im}(U_{\alpha 2} U_{\beta 2}^* U_{\alpha 3} U_{\beta 3}^*)}$$

Or, multiply eqn (2) by $U_{\alpha 3} U_{\beta 3}^*$

$$U_{\alpha 1}^* U_{\beta 1} U_{\alpha 3} U_{\beta 3}^* + U_{\alpha 2}^* U_{\beta 2} U_{\alpha 3} U_{\beta 3}^* + \underbrace{|U_{\alpha 3} U_{\beta 3}|^2}_{\text{REAL !!}} = 0$$

$$\Rightarrow \boxed{\text{Im}(U_{\alpha 1}^* U_{\beta 1} U_{\alpha 3} U_{\beta 3}^*) = - \text{Im}(U_{\alpha 2}^* U_{\beta 2} U_{\alpha 3} U_{\beta 3}^*)}$$

Multiplying through eqn 40 we get (see PD6 2006 page 138)

(iv)

22

$$U = \begin{bmatrix} C_{12} C_{13} & S_{12} C_{13} & S_{13} e^{-i\delta} \\ -S_{12} C_{23} - C_{12} S_{23} S_{13} e^{i\delta} & C_{12} C_{23} - S_{12} S_{23} S_{13} e^{i\delta} & S_{23} C_{13} \\ S_{12} S_{23} - C_{12} C_{23} e^{i\delta} & -C_{12} S_{23} - S_{12} C_{23} S_{13} e^{i\delta} & C_{23} C_{13} \end{bmatrix}$$

Calculate one of these imaginary parts, say

$$\text{Im}(U_{11}^* U_{21} U_{12} U_{22}^*) = C_{12} C_{13} S_{12} C_{13} \text{Im}(U_{21} U_{22}^*)$$

$$= C_{12} C_{13} S_{12} C_{13} (-1) \text{Im}[S_{12} C_{23} + C_{12} S_{23} S_{13} e^{i\delta}] [C_{12} C_{23} - S_{12} S_{23} S_{13} e^{-i\delta}]$$

$$= C_{12} C_{13} S_{12} C_{13} (-1) \left[S_{12}^2 S_{23} S_{13} \overset{C_{23}}{\sin \delta} + C_{12}^2 S_{23} S_{13} C_{23} \sin \delta \right]$$

$$= - \underline{C_{12}^2 C_{13}^2 C_{23} S_{12} S_{13} S_{23} \sin \delta}$$